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Optimum Design of Nonproportionally Damped Structures Using Method of Optimality Criterion

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A structural optimization problem that deals with nonproportional damping is discussed in this paper. The damped motion equation is transformed into a state-space form with displacements and velocities as state variables. The transformed system is in the same form as an undamped eigenvalue problem except that the system matrices are not positive definite. Taking advantage of this familiar eigensystem, the damped eigenvalues and eigenvectors can be obtained using widely available computer codes. The sensitivities for the damped eigenvalues can also be analytically calculated. The structural optimization of the nonproportionally damped system is completed by using the method of optimality criterion. Two numerical examples demonstrate this proposed approach.

Introduction

THE topic of structural optimization subjected to undamped natural frequency constraints has been broadly explored by many researchers.[1–3] As a result, various outcomes have been presented. Some of these methods have been written in computer codes and successfully applied in practical engineering designs.

However, a similar structural optimization problem that deals with nonproportionally damped structures with damped eigenvalue constraints is rarely addressed. In a recent survey paper by Bellos and Inman,[4] less than 40 papers discussing nonproportional damping in the past three decades were cited. The rarity of papers on structural optimization for nonproportionally damped structures is not surprising. Most research into optimum design of nonproportionally damped structures was confined to the extension of the optimum design of the damped vibration absorber initially presented by Ormondryod and Hartog[5] and later extended by Snowdon.[6] A generalized approach that can be applied to a common nonproportionally damped structure has not yet been found. Since the finite element method and various mathematical programming algorithms as well as other methods have demonstrated their abilities in handling undamped structural analyses and optimizations, similar approaches should be applicable to nonproportionally damped structures as well.

It is therefore the purpose of this paper to develop an optimization algorithm which optimizes damped structures subjected to multiple damped eigenvalue constraints. The method proposed in this paper transforms the motion equation into state-space form. The transformed problem is very similar to the eigenvalue problem for an undamped system. The analyses of the eigenvalues and the computations of the damped eigenvalue sensitivities can be followed as for the undamped eigenproblems. The indirect method of optimality criterion[7–9] is employed to perform the structural optimization work.

Theory

The general free motion equation for a viscously damped system is

$$[M]\{\ddot{U}\} + [C]\{\dot{U}\} + [K]\{U\} = 0 \tag{1}$$

where $[M]_{n\times n}$ is the mass matrix of the structure, $[C]_{n\times n}$ is the general viscous damping matrix, $[K]_{n\times n}$ is the stiffness matrix of the structure, $\{U\}$ is the displacement vector, $\{\dot{U}\}$ is the velocity vector, and $\{\ddot{U}\}$ is the acceleration vector.

Since a general damping matrix $[C]$ is usually not diagonalizable by the undamped modal matrix,[10] the widely used modal summation method that decouples Eq. (1) into n independent second-order ordinary differential equations becomes impossible. Following standard procedures,[11] the eigenproblem associated with Eq. (1) can be cast into the following form:

$$[A]\{q_i\} = \lambda_i[B]\{q_i\} \tag{2}$$

where

$$[A] = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \tag{3}$$

$$[B] = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \tag{4}$$

and

$$\{q_i\} = \begin{Bmatrix} u_i \\ \lambda_i u_i \end{Bmatrix} \tag{5}$$

The sensitivity calculations are indispensable to structural optimization. Fox and Kapoor[12] derived an analytical formula to compute the eigenvalue sensitivities for undamped structures. The result is summarized here as

$$\frac{\partial\lambda_i}{\partial\alpha} = \{\phi_i\}^T\left(\frac{\partial[K]}{\partial\alpha} - \lambda_i\frac{\partial[M]}{\partial\alpha}\right)\{\phi_i\} \tag{6}$$

where $[K]$ and $[M]$ are the stiffness and the mass matrices of the undamped structure, respectively; λ_i is the ith undamped eigenvalue of the structure; α is the selected design variable; $\{\phi_i\}$ is the ith orthonormal eigenvector; and the superscript T indicates a transpose of the matrix.

Since Eq. (2) is in the same form as that for the undamped eigenvalue problem, the sensitivity calculations for the damped

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system can be readily obtained by replacing $[K]$, $[M]$ and $\{\phi_i\}$ in Eq. (6) by $[A]$, $[B]$, and $\{q_i\}$, respectively. Equation (7) so obtained is used to find the eigenvalue sensitivities for the damped system:

$$\frac{\partial\lambda_i}{\partial\alpha} = \{q_i\}^T\left(\frac{\partial[A]}{\partial\alpha} - \lambda_i\frac{\partial[B]}{\partial\alpha}\right)\{q_i\} \tag{7}$$

where $\{q_i\}$ is normalized such that $\{q_i\}^T[B]\{q_i\} = 1$.

The sensitivities for the real and the imaginary parts are expressed as

$$\frac{\partial\mathrm{Re}(\lambda_i)}{\partial\alpha} = \mathrm{Re}\left(\frac{\partial\lambda_i}{\partial\alpha}\right) \tag{8}$$

and

$$\frac{\partial\mathrm{Im}(\lambda_i)}{\partial\alpha} = \mathrm{Im}\left(\frac{\partial\lambda_i}{\partial\alpha}\right) \tag{9}$$

where Re and Im stand for taking the real and the imaginary parts of the complex value in the parentheses, respectively.

The method of optimality criterion that requires that the Kuhn-Tucker necessary conditions be satisfied at optimum is chosen to solve the structural optimization problem. The optimization problem is defined as follows:

Minimize

$$W(\alpha) = \sum_{k=1}^{m}\alpha_k w_k \tag{10}$$

subjected to

$$\mathrm{Re}(\bar{\lambda}_i^l) \le \mathrm{Re}(\lambda_i) \le \mathrm{Re}(\bar{\lambda}_i^u) \tag{11}$$

and, for $i = 1, 2, \dots, l$,

$$\mathrm{Im}(\bar{\lambda}_i^l) \le \mathrm{Im}(\lambda_i) \le \mathrm{Im}(\bar{\lambda}_i^u) \tag{12}$$

where the linear objective function W represents the weight of the structure and w_k is the weight associated with design variable α_k; $(\bar{\lambda}_i^l)$ and $(\bar{\lambda}_i^u)$ are the lower and upper bounds of the ith constraint, respectively; l is the number of constraints, and m is the number of design variables.

Equations (11) and (12) show the generalized inequality constraints. To form the Lagrangian function, they will be transformed into standard less than type constraints later. The real and imaginary parts of the eigenvalue are specified by two constraint equations so that the Lagrangian function can be formed without using complex numbers. Besides, the modal damping ratio that plays an important role in the structural response is a function of those two parts. Specifying the limits for the real and imaginary parts not only confines the magnitude of the damped eigenvalue but also tightly constrains the modal damping ratio for the constrained mode in a desired range.

To form the Lagrangian function, the constraint equations (11) and (12) are redefined as the following forms in which each range constraint is expressed by two inequality constraints, for $j = 1, 2, \dots, l$,

$$f_j = \mathrm{Re}(\lambda_j) - \mathrm{Re}(\bar{\lambda}_j^u) \le 0 \tag{13}$$

$$g_j = \mathrm{Re}(\bar{\lambda}_j^l) - \mathrm{Re}(\bar{\lambda}_j) \le 0 \tag{14}$$

$$h_j = \mathrm{Im}(\lambda_j) - \mathrm{Im}(\bar{\lambda}_j^u) \le 0 \tag{15}$$

$$k_j = \mathrm{Im}(\bar{\lambda}_j^l) - \mathrm{Im}(\lambda_j) \le 0 \tag{16}$$

The Lagrangian function is now formed as

$$L(\alpha,\mu) = W(\alpha) + \sum_{j=1}^{l}(\mu_{jf}f_j + \mu_{jg}g_j + \mu_{jh}h_j + \mu_{jk}k_j) \tag{17}$$

where $L(\alpha, \mu)$ is the Lagrangian function; μ_{jf}, μ_{jg}, μ_{jh}, and μ_{jk} are the Lagrange multipliers.

The Kuhn-Tucker necessary conditions for optimality demand

$$\frac{\partial W}{\partial\alpha_k} + \sum_{j=1}^{l}\left(\mu_{jf}\frac{\partial f_j}{\partial\alpha_k} + \mu_{jg}\frac{\partial g_j}{\partial\alpha_k} + \mu_{jh}\frac{\partial h_j}{\partial\alpha_k} + \mu_{jk}\frac{\partial k_j}{\partial\alpha_k}\right) = 0 \tag{18}$$

$$k = 1, 2, \dots, m$$

with

$$\mu_{jf}, \mu_{jg}, \mu_{jh}, \mu_{jk} \ge 0 \tag{19}$$

and, for $j = 1, 2, \dots, l$,

$$\mu_{jf}f_j = \mu_{jg}g_j = \mu_{jh}h_j = \mu_{jk}k_j = 0 \tag{20}$$

Substituting Eqs. (8), (9), and (13)–(16) into Eq. (18) yields

$$w_k + \sum_{j=1}^{l}(\mu_{jf} - \mu_{jg})\frac{\partial\mathrm{Re}(\lambda_j)}{\partial\alpha_k} + (\mu_{jh} - \mu_{jk})\frac{\partial\mathrm{Im}(\lambda_j)}{\partial\alpha_k} = 0 \tag{21}$$

To simplify the mathematical expression, let

$$S_k = \sum_{j=1}^{l}(\mu_{jf} - \mu_{jg})\frac{\partial\mathrm{Re}(\lambda_j)}{\partial\alpha_k} + (\mu_{jh} - \mu_{jk})\frac{\partial\mathrm{Im}(\lambda_j)}{\partial\alpha_k} \tag{22}$$

Equation (21) becomes

$$w_k + S_k = 0 \tag{23}$$

or

$$\frac{-S_k}{w_k} = 1 \tag{24}$$

Multiplying Eq. (24) by $(1 - \beta)\alpha_k$ results in

$$\alpha_k = \left[\beta - (1-\beta)\frac{S_k}{w_k}\right]\alpha_k \tag{25}$$

where β is a relaxation factor for controlling the move limits of design variables in each iteration and its value is between 0 and 1.

The equal sign in Eq. (25) is valid only at the optimality point. Therefore, an iteration loop is created based on this equation. To clarify the use of this equation in an iteration process, it is rewritten in the following form:

$$\hat{\alpha}_k = \left[\beta - (1-\beta)\frac{S_k}{w_k}\right]\alpha_k \tag{26}$$

where $\hat{\alpha}_k$ is the updated kth design variable, and α_k is the current kth design variable.

Table 1 Initial values of design variables and lumped masses of example 1

Design variable no.	K_i, lb/in.	C_i, lb-s/in.
1	7,500	—
2	5,300	—
3	6,100	—
4	4,000	—
5	8,000	—
6	5,000	—
7	11,000	—
8	—	30
9	—	25
10	—	20
11	—	40
12	—	50
13	—	20
14	—	10
Mass	M_i, lb-s^2/in.	
1	10	
2	20	
3	15	
4	10	
5	5	
6	10	
7	1	

Before entering Eq. (26) to find the updated value for α_k, some unknowns in S_k have to be solved first. Those unknowns in S_k are the Lagrange multipliers in Eq. (22). These Lagrange multipliers can be solved as follows.

The damped eigenvalues for the updated structure can be approximately estimated by the first-order Taylor's series expansion, for $i = 1, 2, \ldots, l$,

$$\mathrm{Re}\,(\hat{\lambda}_i) \approx \mathrm{Re}\,(\lambda_i) + \sum_{k=1}^{m} \frac{\partial \mathrm{Re}\,(\lambda_i)}{\partial \alpha_k} \Delta\alpha_k \tag{27}$$

and

$$\mathrm{Im}\,(\hat{\lambda}_i) \approx \mathrm{Im}\,(\lambda_i) + \sum_{k=1}^{m} \frac{\partial \mathrm{Im}\,(\lambda_i)}{\partial \alpha_k} \Delta\alpha_k \tag{28}$$

where $\mathrm{Re}\,(\hat{\lambda}_i)$ and $\mathrm{Im}\,(\hat{\lambda}_i)$ represent the real and imaginary parts of the ith constrained eigenvalue for the updated system, respectively; $\mathrm{Re}\,(\lambda_i)$ and $\mathrm{Im}\,(\lambda_i)$ are the real and imaginary parts of the current ith constrained eigenvalue.

The left-hand sides of these two equations can be replaced by the desired constrained values of the ith constrained eigenvalue, and the $\Delta\alpha_k$ on the right-hand sides, which is the difference of $\hat{\alpha}_k$ and α_k, can be substituted by Eq. (26). Equations (27) and (28) thus become

$$\sum_{k=1}^{m} \frac{\partial \mathrm{Re}\,(\lambda_i)}{\partial \alpha_k} (\beta - 1) \left(1 + \frac{S_k}{w_k}\right) \alpha_k = \mathrm{Re}\,(\bar{\lambda}_i^{u,l}) - \mathrm{Re}\,(\lambda_i) \tag{29}$$

and, for $i = 1, 2, \ldots, l$,

$$\sum_{k=1}^{m} \frac{\partial \mathrm{Im}\,(\lambda_i)}{\partial \alpha_k} (\beta - 1) \left(1 + \frac{S_k}{w_k}\right) \alpha_k = \mathrm{Im}\,(\bar{\lambda}_i^{u,l}) - \mathrm{Im}\,(\lambda_i) \tag{30}$$

where $\mathrm{Re}\,(\bar{\lambda}_i^{u,l})$ and $\mathrm{Im}\,(\bar{\lambda}_i^{u,l})$ stand for the desired values for the ith constraint, and the superscript u, l means the desired value for (λ_i) may either be an upper bound or a lower bound value.

Substituting Eq. (22) into Eqs. (29) and (30) for S_k results in a linear set of equations for the Lagrange multipliers:

$$\sum_{k=1}^{m} \sum_{j=1}^{l} \frac{S_{Rki}}{w_k} \left[\frac{\partial \mathrm{Re}\,(\lambda_j)}{\partial \alpha_k} (\mu_{jf} - \mu_{jg}) + \frac{\partial \mathrm{Im}\,(\lambda_j)}{\partial \alpha_k} (\mu_{jh} - \mu_{jk}) \right] = \mathrm{Re}\,(\bar{\lambda}_i^{u,l}) - \mathrm{Re}\,(\lambda_i) - \sum_{k=1}^{m} S_{Rki} \tag{31}$$

and, for $i = 1, 2, \ldots, l$,

$$\sum_{k=1}^{m} \sum_{j=1}^{l} \frac{S_{Iki}}{w_k} \left[\frac{\partial \mathrm{Re}\,(\lambda_j)}{\partial \alpha_k} (\mu_{jf} - \mu_{jg}) + \frac{\partial \mathrm{Im}\,(\lambda_j)}{\partial \alpha_k} (\mu_{jh} - \mu_{jk}) \right] = \mathrm{Im}\,(\bar{\lambda}_i^{u,l}) - \mathrm{Im}\,(\lambda_i) - \sum_{k=1}^{m} S_{Iki} \tag{32}$$

where

$$S_{Rki} = \frac{\partial \mathrm{Re}\,(\lambda_i)}{\partial \alpha_k} (\beta - 1)\, \alpha_k \tag{33}$$

and

$$S_{Iki} = \frac{\partial \mathrm{Im}\,(\lambda_i)}{\partial \alpha_k} (\beta - 1)\, \alpha_k \tag{34}$$

In Eqs. (31) and (32) the summations are carried out only over the active design variables. If the coefficient matrix formed by Eqs. (31) and (32) for the Lagrange multipliers is not singular, the unique solution for the Lagrange multiplier vector is found by solving these linear equations. The coefficient matrix may become singular under two circumstances. The first possibility is that, during the optimization process, if any of the constrained eigenvalues become overdamped, then the sensitivity for the imaginary part is zero. This zero sensitivity results in zero coefficients in Eq. (32), and therefore a singular matrix is formed. The other possibility of creating a singular matrix is due to the sensitivities of the active design variables being very small or zero. In the iteration process, if one design variable reaches its lower or upper bound, that design variable becomes passive. Only the sensitivities of the active design variables are used to form Eqs. (31) and (32). If these sensitivities happen to be zero or very small, then a singular matrix may occur.

A brief summary for the proposed optimization procedure is depicted by the following steps.

(1) Analyze the initially designed structure for the damped eigenvalues using Eq. (2).

(2) Compute the sensitivity data using Eq. (7).

(3) Evaluate the Lagrange multipliers using Eqs. (31) and (32).

(4) Update the various design variables α using Eq. (26).

(5) Check convergence. If $|\hat{\alpha}_k - \alpha_k| / \alpha_k$ is less than a predetermined value ϵ for all of the design variables, stop. Otherwise, go to step 1.

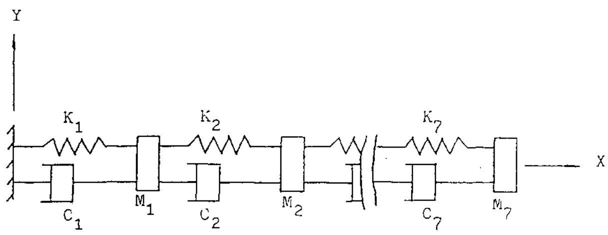

Fig. 1 Spring-mass-damper system.

Numerical Examples

Two numerical examples are employed to demonstrate the effects of the structural optimization using this proposed method. The first example is a seven-degree-of-freedom spring-mass-damper system as shown in Fig. 1. The initial values of the spring constants, the damping coefficients, and the lumped masses are given in Table 1.

The first constraint requires that the real part of the first damped eigenvalue be less than −0.100 and the imaginary part be greater than 8.000. The second constraint demands that the real part of the third damped eigenvalue be less than −0.500 and the imaginary part be greater than 18.000. The weights associated with each spring and damper are assumed to be equal to the values of the spring constant and the damping coefficient of the damper, respectively. That is, the larger the design variable value, the heavier the weight. The side constraints allow each design variable to vary between 0.1 to 4 times of its initially given value. After 11 design iterations, convergence is reached. The final design is shown in Table 2. The design history of the structural weight change and the constrained damped eigenvalues is given in Table 3.

The second numerical example is an in-plane cantilever beam. One translational DOF in the y direction and one rotational DOF in the zz direction are allowed at each free node. Two sets of spring-damper systems are connected between node point 4 and the ground and also between node point 9 and the ground as shown in Fig. 2. The initial values of the design variables k and c are given along with the figure. The Young's modulus is assumed to be 10^7 psi. The mass density is 0.0002588 lb-s^2/in.^{4} The uniform cross-sectional area and the area of moment of inertia are 3.0 in.^{2} and 0.25 in.^{4}, respectively. The behavior constraint is set to have the first damped eigenvalue equal to $-20.0+253.0i$. The weight associated with each design variable equals the value of the spring constant or the damping coefficient accordingly. No side constraints are imposed. Twelve iterations are taken to get the optimum design. Table 4 contains the final design values for the four design variables. Table 5 records the design iteration history.

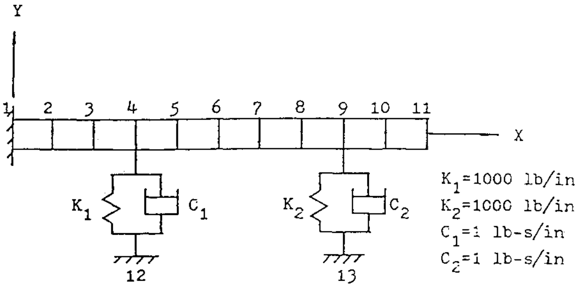

Fig. 2 Cantilever beam with spring-damper absorber.

Table 2 Optimum solution of example 1

Design variable no.	K_i, lb/in.	C_i, lb-s/in.
1	14,369	—
2	13,358	—
3	10,293	—
4	8,002	—
5	7,071	—
6	5,631	—
7	1,100	—
8	—	3.0
9	—	2.5
10	—	2.0
11	—	160.0
12	—	5.0
13	—	2.0
14	—	1.0

Table 3 Iteration history of example 1

Iteration no.[a,b]	Weight, lb	First eigenvalue λ_1	Third eigenvalue λ_3
0*	47,095	$-0.084+5.758i$	$-0.687+14.459i$
1*	58,195	$-0.092+7.399i$	$-0.484+16.521i$
2*	63,878	$-0.100+7.968i$	$-0.502+17.899i$
3*	62,766	$-0.099+7.999i$	$-0.483+18.000i$
4*	61,654	$-0.100+7.999i$	$-0.498+17.997i$
5*	60,870	$-0.100+8.000i$	$-0.497+17.998i$
6*	60,294	$-0.100+8.000i$	$-0.493+17.998i$
7*	59,996	$-0.094+8.000i$	$-0.474+17.987i$
8*	59,990	$-0.100+7.999i$	$-0.501+17.999i$
9	60,000	$-0.100+8.000i$	$-0.500+18.000i$
10	60,000	$-0.100+8.000i$	$-0.500+18.000i$
11	60,000	$-0.100+8.000i$	$-0.500+18.000i$

[a]Iteration 0 represents the initial design.
[b]The superscript * means infeasible solution.

Table 4 Optimum solution of example 2

Design variable no., type	K_i, lb/in. or C_i, lb-s/in.
1, K_1	0.1000
2, K_2	1425.6
3, C_1	0.0001
4, C_2	0.9904

Table 5 Iteration history of example 2

Iteration no.[a,b]	Weight, lb	First eigenvalue λ_1
0*	2002.0	$-22.334+219.87i$
1*	1882.4	$-19.937+249.96i$
2*	1684.1	$-19.854+252.87i$
3*	1564.6	$-19.677+253.03i$
4*	1500.7	$-19.994+252.99i$
5*	1466.8	$-19.998+253.00i$
6*	1448.5	$-19.999+253.00i$
7	1438.6	$-20.000+253.00i$
8	1433.1	$-20.000+253.00i$
9	1430.2	$-20.000+253.00i$
10	1428.6	$-20.000+253.00i$
11	1427.7	$-20.000+253.00i$
12	1427.2	$-20.000+253.00i$

[a]Iteration 0 represents the initial design.
[b]The superscript * means infeasible solution.

The implementation of this proposed approach to a structured existing computer program is quite easy. Since the transformed state-space formulation is in the same form as that for the undamped system, any algorithm which is used to solve the undamped eigenvalue constraint problem can be borrowed to solve this damped problem by simply replacing some matrices with the ones for the damped system.

Concerning the stability of convergence, the relaxation factor β is varied between 0 and 0.9 with increments of 0.1. The two examples are run with these β values. For the first example, when β equals zero, i.e., the largest step size is used, the third eigenvalue becomes overdamped after one iteration. The method fails since a singular matrix is formed by Eqs. (31) and (32). When β equals 0.1, the step size is still too large, and a mild oscillation is observed. The problem has not converged after 50 iterations. For other β values, after one overshooting, the problem converges stably. For β values less than 0.8, the iteration process converges quickly. For the second example, all cases converge stably. But for $\beta = 0.9$, the problem converges slowly since the step size allowed is too small. Based on these two examples, the efficient value for β falls between 0.4 and 0.7.

One drawback of this approach is that the transformation of the problem into state-space form doubles the problem's dimension. The computational efficiency of structural optimization is very important for a large-scale structure. Therefore, an approximate model needs to be used in the reanalysis of the modified structure to increase the efficiency for processing a complex structure. Wang[13] used the complex eigenvectors and their derivatives to form an approximate model. Several test problems showed good approximations using his proposed method. However, sometimes the unstable eigenvalues that have positive real parts would appear. Careful incorporation of any efficient reanalysis method with the currently proposed optimization approach may establish an efficient design optimization process for large-scale nonproportionally damped structures.

Conclusions

An approach that optimizes nonproportionally damped structures using the method of optimality criterion is established in this paper. The analysis of the damped system is indirectly performed in a transformed state-space form that is similar to an undamped eigensystem problem. It is shown that the proposed method is easy to implement. Furthermore, simple numerical examples demonstrate quick and stable convergence.

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References

[1]Khot, N. S., "Optimization of Structures with Multiple Frequency Constraints," *Journal of Computers and Structures,* Vol. 20, No. 5, 1985, pp. 869–876.

[2]Grandhi, R. V., and Venkayya, V. B., "Structural Optimization with Frequency Constraints," *AIAA Journal,* Vol. 26, No. 7, 1988, pp. 858–866.

[3]Wang, B. P., Chang, Y. K., Lawrence, K. L., and Chen, T. Y., "Optimum Design of Structures with Multiple Configurations with Frequency and Displacement Constraints," *Proceedings of the AIAA/ASME/ASCE/AHS/ASC 31st Structures, Structural Dynamics, and Materials Conference,* AIAA, Washington, DC, 1990, pp. 378–384.

[4]Bellos, J., and Inman, D. J., "A Survey on Nonproportional Damping," *The Shock and Vibration Digest,* Vol. 21, No. 10, 1989, pp. 7–12.

[5]Ormondroyd, J., and Hartog, D., "The Theory of the Dynamic Vibration Absorber," *Transactions of the American Society of Mechanical Engineering,* Vol. 50, 1928, pp. A9–A22.

[6]Snowdon, J. C., *Vibration and Shock in Damped Mechanical Systems,* Wiley, New York, 1968.

[7]Arora, J. S., *Introduction to Optimum Design,* 1st ed., McGraw-Hill, New York, 1989, pp. 111–134.

[8]Khot, N. S., "Algorithms Based on Optimality Criteria to Design Minimum Weight Structures," *Journal of Engineering Optimization,* Vol. 5, 1981, pp. 73–90.

[9]Khan, M. R., "Optimality Criterion Techniques Applied to Frames Having General Cross-Sectional Relationships," *AIAA Journal,* Vol. 22, No. 5, 1984, pp. 669–676.

[10]Clough, R. W., and Penzien, J., *Dynamics of Structures,* McGraw-Hill, New York, 1975, pp. 195–198.

[11]Hurty, W. C., and Rubinstein, M. F., *Dynamics of Structures,* Prentice-Hall, Englewood Cliffs, NJ, 1964, pp. 318–322.

[12]Fox, R. L., and Kapoor, M. P., "Rates of Change of Eigenvalues and Eigenvectors," *AIAA Journal,* Vol. 6, No. 12, 1968, pp. 2426–2429.

[13]Wang, B. P., "Efficient Eigensolution Reanalysis of Nonclassically Damped Structures," *Proceedings of 14th National Conference on Theoretical and Applied Mechanics,* (Chung Li, Taiwan), Society of Theoretical and Applied Mechanics of the Republic of China, Taipei, 1990, pp. 363–368.